288. **Proposed by DR. L. E. DICKSON, Associate Professor of Mathematics, The University of Chicago.**

Evaluate the determinant which arises in finding the inverse of the transformation, with binomial coefficients,

$$T: \quad \xi_i=\sum_{j=i}^{g-1}\binom{j}{i}x_j \quad (i=0, 1, \ldots, g-1).$$

Solution by the PROPOSER.

Denote by $D_{n,m}$ the minor of the element in the $(n+1)$th row and $(m+1)$th column. Evidently $D_{nn}=1$, $D_{nm}=0$ for $n<m$. For $n=m+k$, $k>0$,

$$D_{m+k,\,m}=\begin{vmatrix} \binom{m+1}{1} & \binom{m+2}{2} & \binom{m+3}{3}\ldots\binom{m+k}{k} \\ 1 & \binom{m+2}{1} & \binom{m+3}{2}\ldots\binom{m+k}{k-1} \\ 0 & 1 & \binom{m+3}{1}\ldots\binom{m+k}{k-2} \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0\ \ldots\binom{m+k}{1} \end{vmatrix}$$

Studnicka* has evaluated a similar determinant:

$$\begin{vmatrix} \binom{m+1}{1} & \binom{m+1}{2} & \binom{m+1}{3}\ldots\binom{m+1}{k} \\ 1 & \binom{m+1}{1} & \binom{m+1}{2}\ldots\binom{m+1}{k-1} \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0\ \ldots\binom{m+1}{1} \end{vmatrix}=\binom{m+k}{k}.$$

Multiplying the first $j-1$ columns of the latter by, respectively,

$$\binom{j-1}{j-1},\ \binom{j-1}{j-2},\ \binom{j-1}{j-3},\ \ldots,\ \binom{j-1}{1},$$

and add the products to the jth column. Then the ith element in the new jth column is† (for $s=j-r$),

$$\sum_{r=i-1}^{j}\binom{m+1}{r-i+1}\binom{j-1}{j-r}=\sum_{s=0}^{j-i+1}\binom{m+1}{j-i+1-s}\binom{j-1}{s}=\binom{m+j}{j-1+1}.$$

Taking $j=k$, $k-1$, ..., 2, in turn, we obtain $D_{m+k,\,m}$. Hence

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*Cf. Pascal-Leitzmann, *Die Determinanten*, 1900, p. 134.

†Cf. Netto, *Combinatorik*, p. 250, (19); Hagen, *Synopsis*, Vol. 1, p. 65, 5.

$$D_{m+k,\,m}=\binom{m+k}{k},\quad D_{nm}=\binom{n}{m} \text{ if } n>m.$$

Hence the inverse of transformation T is

$$T^{-1}: x_i=\sum_{j=i}^{g-1}(-1)^{i+j}\binom{j}{i}\xi_j \quad (i=0,\,1,\,\ldots,\,g-1).$$

Since the product of the two transformations is the identity, we have

$$\sum_{j=i}^{l}(-1)^{l+j}\binom{l}{j}\binom{j}{i}=\delta_{il} \quad (\delta_{ii}=1,\ \delta_{il}=0 \text{ if } i\neq l).$$

Conversely, from this well known formula (cf. Netto, p. 255, (43)), follows the evaluation of the determinant D.

289. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\left(\frac{n+1}{(n+1)^2-1}+\frac{1}{3}\cdot\frac{n+3}{(n+3)^2-1}+\frac{1}{5}\cdot\frac{n+5}{(n+5)^2-1}+\ldots\right)$ $+\left(\frac{2}{3}\cdot\frac{1}{n+2}+\frac{4}{15}\cdot\frac{1}{n+4}+\frac{6}{35}\cdot\frac{1}{n+6}+\ldots\right)=\frac{n-1}{(n-1)^2-1}+\frac{1}{3}\cdot\frac{n-3}{(n-3)^2-1}$ $+\frac{1}{5}\cdot\frac{n-5}{(n-5)^2-1}+\ldots+\frac{1}{l}\cdot\frac{n-l}{(n-l)^2-1}$, n being any odd integer greater than 1 and $l=n-2$.

Solution by the PROPOSER.

The term by term product of the right hand members of the two Fourier's sine series,

$$1=\frac{4}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots..\right] \text{ and}$$

$$\cos x=\frac{4}{\pi}\left[\tfrac{2}{3}.\sin 2x+\tfrac{4}{15}.\sin 4x+\tfrac{6}{35}.\sin 6x+\ldots..\right]$$

being equal to the product of the left hand members, may be thus written:

$$\cos x=C_0+C_1\cos x+C_2\cos 2x+C_3\cos 3x+\ldots..$$

the C's being constants and functions of the coefficients of the sine terms. But as this product of the right hand members is of the same form as the regular Fourier's cosine series for $\cos x$, it must be identical with the latter series, and, hence, each C except C_1 must be equal to zero. Each C with